THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2020B Advanced Calculus II
Suggested Solutions for Homework 9

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1. Integrate the function $H(x, y, z) = yz$ over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

   Solution. Substituting the equation of the cone $z = \sqrt{x^2 + y^2}$ into the equation of the sphere, we obtain
   $$2(x^2 + y^2) = 4 \Rightarrow x^2 + y^2 = 2$$
   and hence the region of integration Ω is the disk of radius $\sqrt{2}$ in the xy-plane. We will use the level set method in evaluating the integral (similar to Example 4 of Lecture notes 16 from the course website). Let $F(x, y, z) = x^2 + y^2 + z^2 - 4$. Then the gradient is
   $$\nabla F = (2x, 2y, 2z), \|\nabla F\| = (4(x^2 + y^2 + z^2))^{1/2} = (4 \cdot 4)^{1/2} = 4$$
   and so
   $$\|\mathbf{r}_u \times \mathbf{r}_v\| = \frac{\|\nabla F\|}{|F_z|} = \frac{4}{|2z|} = \frac{4}{2z}$$
   since $z \geq 0$ in this case. Therefore the integral is
   $$\begin{aligned}\iint_S H d\sigma &= \iint_\Omega yz\frac{4}{2z}dxdy \\ &= \iint_\Omega 2ydxdy \\ &= \int_0^{2\pi}\int_0^{\sqrt{2}} 2r^2 \sin\theta d\theta dr \\ &= \int_0^{2\pi} \frac{2}{3}r^3 \sin\theta\Big|_{r=0}^{r=\sqrt{2}} d\theta \\ &= -\frac{4\sqrt{2}}{3}\cos\theta\Big|_{\theta=0}^{\theta=2\pi} = 0.\end{aligned}$$
   ◀

2. Compute the flux of the vector field
   $$\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$$
   outward (normal away from the z-axis) through the conical surface S defined by
   $$z = \sqrt{2x^2 + y^2}, \quad 0 \leq z \leq 2.$$

Solution. Using the change of variables $x = \sqrt{2}r\cos\theta, y = 2r\sin\theta$, then $2x^2+y^2 = 4r^2\cos^\theta + 4r^2\sin^2\theta = 4r^4$ for $r \geq 0$ and $\theta \in [0, 2\pi]$. Then on S, $z = \sqrt{2x^2+y^2} = 2r$ and so the range $0 \leq z \leq 2$ gives the range $0 \leq r \leq 1$. Then the parametrization of S in (r, θ) is given by

$$\mathbf{r}(r,\theta) = (\sqrt{2}r\cos\theta, 2r\sin\theta, 2r), \quad 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1$$

with

$$\begin{aligned}
\mathbf{r}_r &= (\sqrt{2}\cos\theta, 2\sin\theta, 2)\\
\mathbf{r}_\theta &= (-\sqrt{2}r\sin\theta, 2r\cos\theta, 0)\\
\mathbf{r}_r \times \mathbf{r}_\theta &= (-4r\cos\theta, -2\sqrt{2}r\sin\theta, 2\sqrt{2}r)
\end{aligned}$$

however, observe that this cross product points inwards towards the z-axis and so we instead take the vector $-(\mathbf{r}_r \times \mathbf{r}_\theta) = (4r\cos\theta, 2\sqrt{2}r\sin\theta, -2\sqrt{2}r)$. Then the flux is

$$\begin{aligned}
\int_S \mathbf{F}\cdot\mathbf{n}d\sigma &= \int_0^{2\pi}\int_0^1 \mathbf{F}(\mathbf{r}(r,\theta)) \cdot -(\mathbf{r}_r \times \mathbf{r}_\theta) dr d\theta\\
&= \int_0^{2\pi}\int_0^1 (4r^2\sin^2\theta, 2\sqrt{2}r^2\cos\theta, -1)\cdot(4r\cos\theta, 2\sqrt{2}r\sin\theta, -2\sqrt{2}r) dr d\theta\\
&= \int_0^{2\pi}\int_0^1 \left(16r^3\sin^2\theta\cos\theta + 8r^3\sin\theta\cos\theta + 2\sqrt{2}r\right) dr d\theta\\
&= \int_0^{2\pi} \left(4r^4\sin^2\theta\cos\theta + 2r^4\sin\theta\cos\theta + \sqrt{2}r^2\right)\Big|_{r=0}^{r=1} d\theta\\
&= \int_0^{2\pi} (4\sin^2\theta\cos\theta + 2\sin\theta\cos\theta + \sqrt{2}) d\theta\\
&= \left(\frac{4}{3}\sin^3\theta + \sin^2\theta + \sqrt{2}\theta\right)\Big|_{\theta=0}^{\theta=2\pi}\\
&= 2\sqrt{2}\pi.
\end{aligned}$$

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3. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x-axis onto the rectangle R_{yz}:

$$1 \leq y \leq 2, \quad 0 \leq z \leq 1$$

in the yz-plane. Let $\mathbf{n}$ be the unit vector normal to S that points away from the yz-plane. Draw a picture of S and find the flux of the field

$$\mathbf{F}(x,y,z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$$

across S in the direction of $\mathbf{n}$.

Solution. The drawing of S and R_{yz} is below.

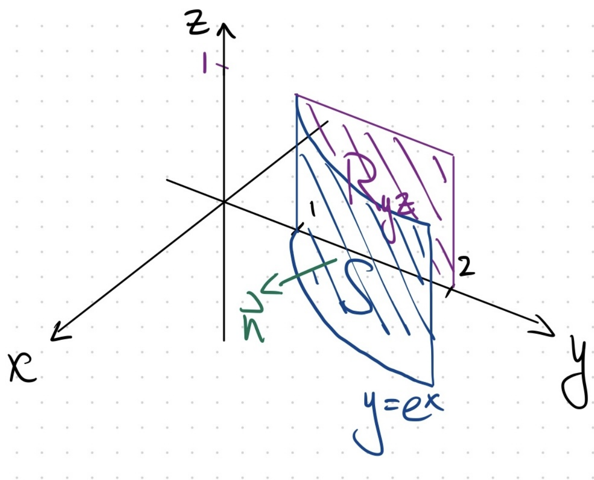

Figure 1: Drawing of S and R_{yz}.

We can parametrize S by $\mathbf{r}(x,z) = (x, e^x, z)$. Then

$$\mathbf{r}_x = (1, e^x, 0), \quad \mathbf{r}_z = (0,0,1),$$
$$\mathbf{r}_x \times \mathbf{r}_z = (e^x, -1, 0), \quad \|\mathbf{r}_x \times \mathbf{r}_z\| = \sqrt{1+e^{2x}}$$

and since for $x \geq 0$ (we are in the first octant), $e^x > 0$ and hence the normal $\mathbf{n} = \dfrac{\mathbf{r}_x \times \mathbf{r}_z}{\|\mathbf{r}_x \times \mathbf{r}_z\|}$ indeed points away from the yz-plane. Then we have that the flux is given by

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma &= \iint_{R_{xz}} \mathbf{F}(\mathbf{r}(x,z)) \cdot (\mathbf{r}_x \times \mathbf{r}_z) dxdz \\
&= \int_0^{\ln(2)} \int_0^1 (-2, 2e^x, z) \cdot (e^x, -1, 0) dzdx \\
&= \int_0^{\ln(2)} \int_0^1 -4e^x dzdx \\
&= \int_0^{\ln(2)} -4e^x dx \\
&= -4e^x \Big|_{x=0}^{x=\ln(2)} \\
&= -4.
\end{aligned}$$

◀

4. Recall that for a parametrized surface $\mathbf{r} : \Omega \to S$ with coordinates (u,v) on Ω, the area is computed by

$$\text{Area}(S) = \iint_\Omega \|\mathbf{r}_u \times \mathbf{r}_v\| dudv.$$

Check that Area(S) is independent of choice of parametrization: Suppose that $\mathbf{g} : \Omega' \to S$ is another parametrization of S with coordinate (s,t) on Ω', then

$$\iint_{\Omega} \|\mathbf{r}_u \times \mathbf{r}_v\| dudv = \iint_{\Omega'} \|\mathbf{g}_r \times \mathbf{g}_t\| dsdt.$$

(Hint: By our standing assumptions on parametrized surfaces, $\mathbf{g}, \mathbf{r}$ are bijective, and $\mathbf{g}^{-1} \circ \mathbf{r}$ and $\mathbf{r}^{-1} \circ \mathbf{g}$ are continuously differentiable. Use change of variable formula for integrals.

Solution. Let $\mathbf{f} = \mathbf{r}^{-1} \circ \mathbf{g} : \Omega' \to \Omega$ be the change of variables (s,t) to (u,v), so $(u,v) = f(s,t)$. Since $\mathbf{g} = \mathbf{r} \circ \mathbf{f}$, we have $\mathbf{g}(s,t) = \mathbf{r}(u,v)$. Then by the chain rule, we have

$$\begin{aligned} \mathbf{g}_s &= \mathbf{r}_u \frac{\partial u}{\partial s} + \mathbf{r}_v \frac{\partial v}{\partial s} \\ \mathbf{g}_t &= \mathbf{r}_u \frac{\partial u}{\partial t} + \mathbf{r}_v \frac{\partial v}{\partial t}. \end{aligned}$$

Then by linearity of the cross product, we have

$$\begin{aligned} \mathbf{g}_s \times \mathbf{g}_t &= \left(\mathbf{r}_u \frac{\partial u}{\partial s} + \mathbf{r}_v \frac{\partial v}{\partial s} \right) \times \left(\mathbf{r}_u \frac{\partial u}{\partial t} + \mathbf{r}_v \frac{\partial v}{\partial t} \right) \\ &= \mathbf{r}_u \frac{\partial u}{\partial s} \times \mathbf{r}_u \frac{\partial u}{\partial t} + \mathbf{r}_u \frac{\partial u}{\partial s} \times \mathbf{r}_v \frac{\partial v}{\partial t} \\ &\quad + \mathbf{r}_v \frac{\partial v}{\partial s} \times \mathbf{r}_u \frac{\partial u}{\partial t} + \mathbf{r}_v \frac{\partial v}{\partial s} \times \mathbf{r}_v \frac{\partial v}{\partial t} \\ &= (\mathbf{r}_u \times \mathbf{r}_v) \left(\frac{\partial u}{\partial s} \frac{\partial v}{\partial t} \right) + (\mathbf{r}_u \times \mathbf{r}_v) \left(-\frac{\partial v}{\partial s} \frac{\partial u}{\partial t} \right) \\ &= (\mathbf{r}_u \times \mathbf{r}_v) \det \begin{bmatrix} \frac{\partial u}{\partial s} & \frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial s} & \frac{\partial v}{\partial t} \end{bmatrix} \\ &= (\mathbf{r}_u \times \mathbf{r}_v) \det J_{\mathbf{f}} \end{aligned}$$

where $J_{\mathbf{f}}$ is the change of variables matrix. Then taking norms we have

$$\|\mathbf{g}_s \times \mathbf{g}_t\| = \|\mathbf{r}_u \times \mathbf{r}_v\| |\det J_{\mathbf{f}}|.$$

Finally, we see that

$$\begin{aligned} \iint_{\Omega'} \|\mathbf{g}_s \times \mathbf{g}_t\| dsdt &= \iint_{\Omega'} \|\mathbf{r}_u \times \mathbf{r}_v\| |\det J_{\mathbf{f}}| dsdt \\ &= \iint_{\Omega} \|\mathbf{r}_u \times \mathbf{r}_v\| dudv \end{aligned}$$

where we used the change of variables formula in the last line above. ◀

5. Verify Stokes' Theorem for the vector field

$$\mathbf{F} = 2xy\mathbf{i} + x\mathbf{j} + (y+z)\mathbf{k}$$

and the surface S defined by $z = 4 - x^2 - y^2$, $z \geq 0$, oriented with the unit normal $\mathbf{n}$ pointing upward. (Compute both sides of Stokes' theorem directly and check that they are equal.)

Solution. For the left-hand side, the boundary C of S is given by $0 = 4 - x^2 - y^2$, i.e., the circle of radius 2. We parametrize C by

$$\mathbf{r}(\theta) = (2\cos\theta, 2\sin\theta, 0), \quad \theta \in [0, 2\pi]$$

with

$$\mathbf{r}'(\theta) = (-2\sin\theta, 2\cos\theta, 0).$$

So the contour integral is

$$\begin{aligned}
\oint_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(\theta)) \cdot \mathbf{r}'(\theta) d\theta \\
&= \int_0^{2\pi} (8\cos\theta\sin\theta, 2\cos\theta, 2\sin\theta) \cdot (-2\sin\theta, 2\cos\theta, 0) d\theta \\
&= \int_0^{2\pi} (-16\cos\theta\sin^2\theta + 4\cos^2\theta) d\theta \\
&= -\frac{16}{3}\sin^3\theta \Big|_{\theta=0}^{\theta=2\pi} + 4\left(\frac{\theta}{2} + \sin\theta\cos\theta\right)\Big|_{\theta=0}^{\theta=2\pi} \\
&= 4\pi.
\end{aligned}$$

For the right-hand side, we first compute the flux of F. We have

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} \\
&= \left(\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y+z \end{vmatrix}, - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ 2xy & y+z \end{vmatrix}, \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy & x \end{vmatrix} \right) \\
&= (1, 0, 1 - 2x).
\end{aligned}$$

To parametrize the surface S, we use cylindrical coordinates. Let $x = r\cos\theta, y = r\sin\theta, \theta \in [0, 2\pi]$. Then the equation $z = 4 - x^2 - y^2 = 4 - r^2$ and the range $z \geq 0$ becomes $0 \leq r \leq 2$. Then the surface S is given by the parametrization

$$\begin{aligned}
\mathbf{r}(r, \theta) &= (r\cos\theta, r\sin\theta, 4 - r^2) \\
\mathbf{r}_r &= (\cos\theta, \sin\theta, -2r) \\
\mathbf{r}_\theta &= (-r\sin\theta, r\cos\theta, 0) \\
\mathbf{r}_r \times \mathbf{r}_\theta &= (2r^2\cos\theta, 2r^2\sin\theta, r)
\end{aligned}$$

where the cross product is pointing upward since $r \geq 0$. Then the surface integral is

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma &= \int_0^{2\pi} \int_0^2 (\nabla \times \mathbf{F})(\mathbf{r}(r, \theta)) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta) dr d\theta \\
&= \int_0^{2\pi} \int_0^2 (1, 0, 1 - 2r\cos\theta) \cdot (2r^2\cos\theta, 2r^2\sin\theta, r) dr d\theta \\
&= \int_0^{2\pi} \int_0^2 r dr d\theta \\
&= \int_0^{2\pi} \frac{1}{2} r^2 \Big|_{r=0}^{r=2} d\theta \\
&= 4\pi
\end{aligned}$$

which we verify matches the value of the integral on the left-hand side. ◀